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RECONSTRUCTION OF THE PHASE SPECTRUM OF ASTRONOMICAL IMAGES ON THE BASIS OF BISPECTRAL ANALYSIS OF SPECKLE INTERFEROGRAMS

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ABSTRACT. *A technique for obtaining the phase Fourier spectrum of object images, based on bispectral analysis of speckle interferograms, is described. It is shown that in the case of non-uniform detector sensitivity over the field the functions necessary for elimination of the photon bias in the bispectrum differ from the power spectra. An iteration algorithm is proposed for reconstructing the phase spectrum from the bispectrum that makes the best use of the redundant bispectrum to increase the signal-to-noise ratio in the reconstructed Fourier spectrum of an object.*

INTRODUCTION

The "phase problem" in classical speckle interferometry (Fienup, 1978) can be solved if instead of the power spectrum one uses the speckle images function, which, when averaged over all short-exposure images, preserves information not only on the modulus but also on the Fourier spectrum phase of the image. The selected function must at high spatial frequencies have a high signal-to-noise ratio, and be easily calculated. One such function is the bispectrum, the product of three Fourier spectrum images expressed in different coordinates:

$$I^{(3)}(\vec{u},\vec{v})=I(\vec{u})I(\vec{v})I(-\vec{u}-\vec{v}), \quad (1)$$

where $I(\vec{u})$ is the Fourier spectrum of the image, and $\vec{u}$, $\vec{v}$, $-\vec{u}-\vec{v}$ are the coordinates in the Fourier region in which the spectrum is expressed (Nikias and Raguver, 1987).

Bispectral analysis was first used in astronomy to analyse interplanetary glow in the metre radio wavelength range. Readhead et al. (1977) showed the analogy of this technique to the closure phase at frequencies higher than that of the atmosphere cut-off, and that this method is restricted not only to radio frequencies but can be applied to any system that constructs interference images in sufficiently short times. Therefore bispectral analysis may be directly applied to speckle interferograms to measure both the modulus and phase of the diffraction-limited Fourier spectrum of an object image. The use of bispectrum analysis in speckle interferometry makes it possible to remove telescopic aberrations and atmospheric distortions and obtain a direct image with a high, close to diffraction, angular resolution (Lohmann et al., 1983). In other words bispectral analysis is the solution to the phase problem in speckle interferometry. Although there is no theory universal enough to provide a full idea of the capabilities of bispectral analysis, the methods developed on its basis show good results when applied to specific problems.

Let us enumerate the basic properties of the bispectrum. As a result of the commutative character of multiplication the rearrangement of spectra does not change the bispectrum value. Besides this, if the image is real its spectrum is then symmetrical about the conjugation, i.e. $I(\vec{u},)=I^{*}(-\vec{u})$, so the bispectrum is also symmetrical $I^{(3)}(\vec{u},\vec{v})=I^{(3)*}(-\vec{u},-\vec{v})$. Using these properties, we can readily derive twelve symmetries of the bispectrum:

$$
\begin{aligned}
T_{01}&: (\vec{u},\vec{v}) \Rightarrow (\vec{u},\vec{v})\\
T_{02}&: (\vec{u},\vec{v}) \Rightarrow (\vec{v},\vec{u})\\
T_{03}&: (\vec{u},\vec{v}) \Rightarrow (\vec{u},-\vec{u}-\vec{v})\\
T_{04}&: (\vec{u},\vec{v}) \Rightarrow (\vec{v},-\vec{u}-\vec{v})\\
T_{05}&: (\vec{u},\vec{v}) \Rightarrow (-\vec{u}-\vec{v},\vec{v})\\
T_{06}&: (\vec{u},\vec{v}) \Rightarrow (-\vec{u}-\vec{v},\vec{u})\\
T_{07}&: (\vec{u},\vec{v}) \Rightarrow (-\vec{u},-\vec{v})\\
T_{08}&: (\vec{u},\vec{v}) \Rightarrow (-\vec{v},-\vec{u})\\
T_{09}&: (\vec{u},\vec{v}) \Rightarrow (-\vec{u},\vec{u}+\vec{v})\\
T_{10}&: (\vec{u},\vec{v}) \Rightarrow (-\vec{v},\vec{u}+\vec{v})\\
T_{11}&: (\vec{u},\vec{v}) \Rightarrow (\vec{u}+\vec{v},-\vec{v})\\
T_{12}&: (\vec{u},\vec{v}) \Rightarrow (\vec{u}+\vec{v},-\vec{u}).
\end{aligned}
\tag{2}
$$

Analysis of the symmetries has revealed a number of fundamental properties of the bispectrum associated with its invariance:

- the bispectrum is invariant with respect to the image displacement;
- the zero-asymmetrical signals can not be recovered from the bispectrum;
- the bispectrum does not change when multiplying the Fourier spectrum of an image by

any exponential factor $\exp(\beta \cdot u)$, where β is the complex constant.

Also, the bispectral approach to the problem of image reconstruction allows us to:

- gain information related to deviations from the normality;
- directly estimate the Fourier spectrum phases of an object image;
- reveal non-linear mechanisms to form an image.

Finally, bispectral analysis is low-sensitive to the additive noise, while the signal-to-noise ratio and the limiting magnitude are the same as in speckle interferometry (Lohmann & Wirnitzer, 1984).

On the other hand, there are difficulties in applying the bispectrum associated with the necessity, due to its 4-dimensionality, of working with large files of digits.

THE BISPECTRUM OF SPECKLE INTERFEROGRAMS

A two-dimensional intensity distribution of the n-th speckle interferogram can be written on the basis of the coherent, quasimonochromatic, spatially-invariant equation of the image:

$$i_n(\vec{x})=o(\vec{x})*p_n(\vec{x}), \tag{3}$$

where $\vec{x}$ is the two-dimensional vector, $o(\vec{x})$ is the two-dimensional intensity distribution of an object, * describes the convolution operation, and $p_n(\vec{x})$ is the function of point scattering. Using the convolution theorem we derive an expression for the averaged bispectrum of speckle interferograms:

$$\langle I_n^{(3)}(\vec{u},\vec{v})\rangle = J^{(3)}(\vec{u},\vec{v}) \cdot \langle P_n^{(3)}(\vec{u},\vec{v})\rangle, \tag{4}$$

where $O^{(3)}(\vec{u},\vec{v})$ is the bispectrum of an object image, and $\langle P_n^{(3)}(\vec{u},\vec{v})\rangle$ is the bispectrum optical transfer function (BOTF) of the system "atmosphere + telescope". Theoretical and experimental investigations of BOTF (Lohmann et al., 1983) have shown that it does not depend on aberrations of the telescope, and is real and positive at all frequencies up to the cut-off frequency of the telescope. Owing to this the complex bispectrum phase of an object is equivalent to the bispectrum phase of interferograms, i.e.

$$phase\ \{J^{(3)}(\vec{u},\vec{v})\} = phase\ \{\langle I_n^{(3)}(\vec{u},\vec{v})\rangle\}. \tag{5}$$

The Fourier phase of the object image can be directly obtained from the averaged bispectrum of speckle interferograms without compensation for BOTF. This circumstance allows us to avoid distortions that arise when dividing spectra.

Using the bispectrum symmetries and the finiteness of the region of determination of the Fourier spectrum, the volume of the computed part of the bispectrum can be considerably reduced. Pehlemann and von der Luhe (1989) have shown that to reconstruct an image of the size $N\times N$ it is necessary to calculate the

$$3N^4/64+3N^3/32+N^2/2+N/2+1 \tag{6}$$

number of bispectrum elements, which totals 98 Mbytes for an image of 128×128 and 6.2 Mbytes for 64×64. Here the image size implies either the Fourier spectrum image size or the image size expressed in resolution elements of the system forming the image. Therefore the number of elements defined by (6) is essentially overestimated.

Although bispectrum analysis makes it possible to reconstruct both components of the Fourier spectrum, in practice it is applied more frequently to recovery of the Fourier phase alone using the relationship:

$$exp\{\iota\beta(\vec{u},\vec{v})\}=exp\{\iota(\theta(\vec{u})+\theta(\vec{v})-\theta(\vec{u}+\vec{v}))\}, \tag{7}$$

where $\theta(\vec{u})$ is the phase of the object, and $\beta(\vec{u},\vec{v})$ is the phase of the bispectrum. Wirnitzer (1985) has shown that the application of bispectral analysis for reconstructing the phase spectrum is most expedient since the signal-to-noise ratio is at its maximum here. For reconstruction of the Fourier modulus it is advisable to use classical speckle interferometry since this procedure is better developed, and one element of the power spectrum has higher signal-to-noise ratio than one element of the bispectrum.

CALCULATION OF THE BISPECTRUM

If the signal-to-noise ratio in an individual speckle interferogram is not large then it is preferable to compute in the Fourier space the entire selected part of the bispectrum at once. Technical aspects of selecting the bispectrum part were stated by Pehlemann and von der Luhe (1989). In that paper it is shown that as the nonredundant part of the bispectrum one may choose that part of it which is determined by the following relations:

$\{(i,j,k,l)\,|$

$i=-N/2,\ldots,0;$

$k=-N/4,\ldots,0;$

$k\geqslant i;$

$i+k\geqslant -N/2;$

$j,l,j+l$ with $\{-N/2,\ldots,N/2-1\};$

$l\geqslant j$ if $k=i;$

$l\leqslant 0$ if $k=0\}.$

Here, i, j, k, l denote integer spatial frequency indices, for the N×N matrix of pixels of the discrete Fourier spectrum. The total number of elements in conditions

(8) is defined by expression (6). The part of the bispectrum defined by (8) contains all the information available in the whole bispectrum. Any further decrease in the calculated part of the bispectrum is accompanied by information loss. If the size of the bispectrum part determined by expression (6) exceeds the internal storage of the computer then it is better to reduce the size of the bispectrum but not compute it in parts. In case the excess is considerable, the size of the bispectrum can be reduced by setting limits (Hofman and Weigelt, 1986): $|k|$, $|l|<M$, where M is a parameter ($M<N/2$). By selecting M one may choose an optimum part of the bispectrum. If the bispectrum only slightly exceeds the storage of the computer then in the selection of j and l with different signs the bispectrum decreases by a factor of 3, but if they are selected to be of the same sign the bispectrum decreases by 1/3. Since we only use the bispectrum for reconstructing the phase spectrum, we do not compute elements which contain no phase information. These are the elements for which $i=j=0$ or $k=l=0$.

In speckle interferometry of astronomical objects fainter than 8^m at the prime focus of the 6 m telescope the quantum nature of light begins to manifest itself. An individual speckle interferogram of such an object is a group of registered photons. If the spatial nonuniformity of the detector sensitivity can be neglected then the intensity distribution in an individual speckle interferogram can be represented by the sum of delta functions;

$$d(\vec{x})=\sum_{j=1}^{N}\delta(\vec{x}-\vec{x}_j), \qquad (9)$$

where N is the number of photons registered in the n-th speckle image, $\vec{x}$ is the radius-vector of the j-th photon, and $\delta(\vec{x})$ is the two-dimensional Dirac delta-function. The simple substitution of (9) into the expression for the bispectrum yields their biased estimates. This bias is associated with the correlation of photons with themselves (Goodman and Belsher, 1976). To cancel the bias, it is necessary to exclude all possible combinations at which such correlations arise. If the non-uniformity of the detector sensitivity can not be ignored, then, taking into account the weight correcting matrix, the intensity distribution in a speckle interferogram is written as the sum of delta-functions and weights:

$$d(\vec{x})=\sum_{j=1}^{N}q(\vec{x}')\cdot\delta(\vec{x}-\vec{x}_j), \qquad (10)$$

where $q(\vec{x})$ is the matrix of weight coefficients, $\delta(\vec{x})$ is the two-dimensional Dirac delta-function, $\vec{x}_i$ is the radius-vector of the i-th photon in the frame, and N is the number of registered photons in the frame. To remove the photon bias arising due to the correlation of the photon with itself, concurrently with the bispectrum it is necessary to calculate the functions determined by the conditions:

a) $j=l=k$:

$$\langle FT\left\{\sum_{j=1}^{N} q^3(\vec{x}_j)\cdot\delta(\vec{x})\right\}\rangle = const; \qquad (11)$$

b) $j=l\neq k$:

$$\langle FT\left\{\sum_{l=1}^{N}\sum_{k=1}^{N} q^2(\vec{x}_l)\cdot q(\vec{x}_k)\cdot\delta(\vec{x}-\vec{x}_k+\vec{x}_l)\right\}\rangle = const \qquad (12)$$

c) $j=k\neq l$,

$$\langle FT\left\{\sum_{l=1}^{N}\sum_{k=1}^{N} q(\vec{x}_l)\cdot q^2(\vec{x}_k)\cdot exp\{2\pi i[(\vec{V}\cdot\vec{x}_l)-(\vec{V}\cdot\vec{x}_k)]\}\cdot\delta(\vec{x}-\vec{x}_k+\vec{x}_l)\right\}\rangle = const \qquad (13)$$

d) $j\neq l=k$.

$$\langle FT\left\{\sum_{j=1}^{N} q(\vec{x}_j)\cdot exp\{-2\pi i(\vec{V}\cdot\vec{x}_j)\}\cdot\sum_{k=1}^{N} q^2(\vec{x}_k)\cdot exp\{2\pi i(\vec{V}\cdot\vec{x}_k)\}\cdot\delta(\vec{x})\right\}\rangle = const. \qquad (14)$$

The obtained result agrees with the one described by Winitzer (1985) if the matrix of weight coefficients is identical to unity. In the general case the photon bias function is a complex function that can be written as:

$$S(\vec{u})+S(-\vec{u}-\vec{v})+S(\vec{v})-2\cdot const, \qquad (15)$$

where:

$$S(\vec{u})=FT\left\{\langle\sum_{l=1}^{N}\sum_{k=1}^{N}q^2(\vec{x}_l)\cdot q(\vec{x}_k)\cdot\delta(\vec{x}-\vec{x}_k+\vec{x}_l)\rangle\right\}. \qquad (16)$$

Thus, to obtain an unbiased estimate of the bispectrum, it is necessary to calculate the function $S(\vec{u})$ and the constant (14) except for the selected part of the bispectrum. The unbiased estimate of the bispectrum has the form:

$$I^{(3)}(\vec{u},\vec{v})=\langle D(\vec{u})D(\vec{v})D(-\vec{u}-\vec{v})\rangle-S(\vec{u})-S(\vec{v})-S(-\vec{u}-\vec{v})+2\ const, \qquad (17)$$

where

$$D(\vec{u})=\sum_{j=1}^{N} d(\vec{x}_j)\ exp\{-2\pi i(\vec{u}\cdot\vec{x}_j)\} \qquad (18)$$

is a Fourier trasform from $d(\vec{x})$.

PHASE RECOVERY FROM THE 4-DIMENSIONAL BISPECTRUM

The relation between the bispectrum phase and the Fourier spectrum phase can be written as a set of equations:

$$exp\{\iota[\varphi(m,n)+\varphi(k,l)-\varphi(m+k,n+l)]\}=exp\{\iota\beta(m,n,k,l)\}, \qquad (19)$$

where k, l, m, n satisfy conditions (8). To solve this essentially redundant set of equations one can either apply the recurrent algorithm (Bartelt et al., 1984) or solve it by the least-squares method. In the former case the phase spectrum is calculated by the following recurrent relation:

$$exp\{\iota\varphi(i,j)\}= const \sum_{k=1}^{p}\sum_{l=m}^{n} exp\{\iota[\varphi(k,l)+\varphi(i-k,j-l)-\beta(i-k,j-l,k,l)]\}, \qquad (20)$$

$$m = \begin{cases} -N/2 & \text{if } j<0 \\ j-N/2 & \text{if } j>0 \\ j & \text{if } i=k \text{ and } j<0 \end{cases}, \qquad n = \begin{cases} j+N/2 & \text{if } j<0 \\ N/2 & \text{if } j>0 \\ j & \text{if } i=k \text{ and } j>0 \end{cases}, \qquad (21)$$

if $i=0$

$$exp\{\iota\varphi(0,j)\}= const \sum_{l=1}^{p} exp\{\iota[\varphi(0,l)+\varphi(0,j-l)-\beta(0,j-l,0,l)]\}, \qquad (22)$$

where p is the integer part of $j/2$. To realize the recurrent algorithm initial conditions are needed, namely, the phase values at three points $\varphi(0,0)$, $\varphi(0,1)$, $\varphi(1,0)$. The phase value at (0,0) may be assumed to be zero since for real images the phase at zero always takes on the values $\pm\pi M$, where M is any integer. The values of $\varphi(0,1)$ and $\varphi(1,0)$ can be calculated by the formula

$$\varphi(0,1)=2\sum_{i=2}^{N/2}\beta(0,i-1,0,1)/N, \qquad (23)$$

where N is the number of spectrum elements along the Y axis. A similar formula is used for $\varphi(1,0)$. One can achieve a considerable increase in the signal-to-noise ratio in the reconstructed phase spectrum if the summation in (20) and (22) is done with the weights:

$$exp\{\iota\varphi(i,j)\}= const \sum_{k=1}^{p}\sum_{l=m}^{n} W(i-k,j-l,k,l)\ exp\{\iota[\varphi(k,l)+\varphi(i-k,j-l)-\beta(i-k,j-l,k,l)]\}.$$

Values proportional to the signal-to-noise ratio of the combined spectrum are chosen as weight coefficients $W(i, j, k, l)$:

$$exp\{\iota[\varphi(k,l)+\varphi(i,j)-\beta(i,j,k,l)]\}. \qquad (25)$$

Shortcomings in the recurrent algorithm are the facts that the error of determination extends from low to high frequencies and is permanently accumulated. Spectrum points with a modulus value close to zero are especially important since at these points the phase may have an uncertain value, which, when used recurrently, distorts the phase spectrum. When averaging, these points must have zero weight. When solving the redundant system (19) by the least-squares method one can avoid extending the error at high frequencies by making full use of the bispectrum's redundance. In the iteration algorithm that we have developed the whole phase spectrum below the cut-off frequency of the telescope is estimated at each iteration. The basic idea consists of reconstructing the phase spectrum $\varphi(k,l)$ for which the expression

$$\sum_k \sum_l \sum_i \sum_j \{[\varphi(k,l)+\varphi(i,j)-\varphi(i+k,j+l)-\beta(i,j,k,l)]/\sigma_\beta(i,j,k,l)\}^2 \tag{26}$$

has a minimum, where $\sigma_\beta(i,j,k,l)$ is the standard bias of the bispectrum phase. As a zero approximation we use the phase spectrum obtained by the recurrent procedure. To obtain a further approximation, we use the following algorithm:

- from the files are selected: the element of the bispectrum phase $\beta(i,j,k,l)$, its standard bias $\sigma_\beta(i,j,k,l)$, and the elements of the Fourier phase $\varphi_{m-1}(i,j)$, $\varphi_{m-1}(k,l)$ and $\varphi_{m-1}(i+k,j+l)$ derived by the preceding iteration;
- new values of the phase $\varphi_m(i,j)$, $\varphi_m(k,l)$ and $\varphi_m(i+k,j+l)$ are calculated using the following relations:

$$exp\{\iota\varphi_m(i+k,j+l)\}=exp\{\iota[\varphi_{m-1}(i,j)+\varphi_{m-1}(k\ l)-\beta\ (i\ j\ k\ l)]\},$$

$$exp\{\iota\varphi_m(i,j)\}=exp\{\iota[\varphi_{m-1}(i+k,j+l)-\varphi_{m-1}(k\ l)+\beta\ (i\ j\ k\ l)]\},$$

$$exp\{\iota\varphi_m(k,l)\}=exp\{\iota[\varphi_{m-1}(i+k,j+l)-\varphi_{m-1}(i,j)+\beta\ (i,j,k,l)]\}; \tag{27}$$

- new values are summed with the weights $1/\sigma_\beta(i,j,k,l)$ in a file specially allocated to a new iteration;
- the procedure is completed for all elements of the bispectrum;
- the averaged phase values are taken as the *m*-th approximation.

The values of expression (26) are selected as the criterion. As a rule, after 10-15 iterations the value of the expression is little affected.

In conclusion it should be noted that the algorithm for the bispectral analysis of speckle interferograms described in the paper is not unique. Algorithms that use the bispectrum properties more efficiently are also possible. Nevertheless, reconstruction of the phase spectrum by the least-squares method most effectively exploits the bispectrum's redundance to increase the signal-to-noise ratio, and the use of the described algorithm for reconstructing astronomical images makes it possible to study targets with a high spatial resolution.

REFERENCES

Bartelt H., Lohmann A., Wirnitzer B.: 1984, *Appl. Opt.* **23**, 3121.

Wirnitzer B.: 1985, *J. Opt. Soc. Am. A2*, No. 1, 14.

Goodman J. W., Belsher J.: 1976, *Proc. Soc. Photo-Opt. Instrum. Eng.*, **75**, 141.

Lohmann A., Weigelt G., Wirnitzer B.: 1983, *Appl. Opt.*, **22**, 4028.

Lohmann A., Wirnitzer B.: 1984, *Proc., IEEE*, **22**, 889.

Nikias H.L., Raguver M.P.: 1987, *Proc. IEEE (in Russian)*, **75**, 7, 5.

Pehlemann E., von der Luhe O.: 1989, *Astron. Astrophys.*, **216**, 337.

Readhead A.C.S., Wilinson P.N., Purcell G.H.: 1977, *Astrophys. J.*, **215**, No. 2, L13-L15.

Fienup J.R.: 1978, *Opt. Lett.*, **3**, 27.

Hofman K.-H., Weigelt G.: 1986, *Appl. Opt.*, **25**, 4280.

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